PREDICTABILITY AND TRAPPING UNDER CONDITIONS OF GLOBALIZATION OF AGRICULTURAL TRADE: AN APPLICATION OF THE CDS APPROACH

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Executive Summary

The purpose of the paper is to demonstrate how timely investments in the ecosphere promote high level stable agricultural income. It is shown that the recovery capability of the ecosphere together with strong agricultural productivity relative to the rest of the economy may be considered as control parameters of the system. They play a crucial role for agriculture to escape from a low level trap. The scenario leading to the escape is based on an economic strategy leading to a sequence of local and global bifurcations, changing the dynamical structure of the system. The effects of globalization on the system's dynamics and stable sets are considered. Globalization is taken into account by means of non-autonomous impulses making selected system parameters oscillate. Necessary conditions for sustainable agriculture development are established.

Key Words: sustainable agriculture; ecosphere; dynamics; low level trap; attractors; bifurcations; globalization; autonomous systems.

Contents

I. Introduction

Natural resource based economies are notoriously prone to wide fluctuation in performance, whether that be measured in terms of incomes, employment, efficiency or reinvestment. This paper builds on previous work to demonstrate that this vulnerability may be addressed by strategic investments selected by understanding the complex dynamics of the relationships between the natural resource systems with the rest of the economy and the environment.

Vulnerability to fluctuation is a serious issue. The rural economy takes the first shock in the form of boom and bust incomes, with consequent human anxiety and concentration of property rights. Beyond some intensity of fluctuation, national economies with prominent natural resource sectors, are buffeted by currency devaluation, trade irritation, costly regional and sectoral expenditure on safety nets, and sacrifice of third tier human rights. These consequences have been experienced within Canada in 1998.

Our model explains economic fluctuations in terms of relationships among agriculture, industry and the ecosphere. The model is robust enough to allow the relationships to be played out in a cooperative, competitive or predatory manner, depending on the management strategy. The type of behaviour is modeled by changing the values of structural parameters in the system.

The principal focus in this paper is on how to achieve sustained higher levels of performance for agriculture, defined in terms of smaller more predictable fluctuation. The problem may be characterized as a trapping for which the first step is to ensure that none of the three systems are trapped at low (unsatisfactory) levels of performance. The conditions for escape are examined, which can lead to the desired higher levels of performance. Sustainability could be interpreted as the process of escape and subsequent establishment of new conditions for trapping at higher levels, pending the need for subsequent growth. Consequently, sustained development may be characterized as an eternal process of learning about relationships, the subject of this paper.

This paper finds that recovery capability for the economic and environmental systems is a particularly powerful management instrument for achieving sustainability. This capability is embedded within the long term strategic features of their relationships one to the other. Recovery is examined in the business sense of structural resiliency to periods of low income, and in the environmental sense of revitalization of attributes useful to agriculture. The environment, hereafter referred to as the ecosphere, is modeled to recover naturally and with reinvestment of revenue by agriculture to enhance recovery through investment in conservation, restoration and rehabilitation.

The investigative method employed here uses *ceteris paribus* numerical experiments based on the analysis of a complex dynamical model. The analysis of the model reported here, ensures persistence and the possibility of bifurcation. Parameters are held constant at initial values chosen to demonstrate the long term consequences of an investment strategy with all other factors held constant. Impulsive changes in selected parameters are used to test the effects of global change during an investment program.

The principles of equilibrium, stability, persistence, attraction and bifurcation are explained in comfortable language for the interdisciplinary reader. Each result is illustrated with graphs of phase portraits and time series of outcomes.

The problems are solved within the framework of a revised model of the stabilized ecosphere. This model is a modification of the basic model proposed by Apedaile et al. (1996). The cornerstone of this new model is investments by the agricultural sector to stabilize the ecosphere.

In the next section we review our previous model and introduce the revised model, justifying its construction. Basic mathematical properties of the revised model as well as possible bifurcations are described in Section III. We also show how the ecosphere may be stabilized and explain the consequences of such stabilization. It is shown that insufficient investments in the ecosphere destabilize outcomes and lead to the low level trapping of agricultural income. Section IV uses these features of the model to develop a scenario for escape from such traps.

In Section V globalization effects are represented as oscillations of the system's structural parameters. It is shown that agricultural incomes may be kept at relatively high levels in spite of considerable oscillations of agricultural productivity, if ecospheric recovery is sufficiently fast. Oscillation of terms of trade may result in the appearance of strange attractors. The latter outcome affects the system's predictability and insurability.

II. Model of the Stabilized Ecosphere – Revised Basic Model

1. Previous Models

The first version of the model, describing the interactions between agriculture and related industries in a system of rural economy, proposed by Apedaile et al., 1994. Agricultural (A) and industrial (I) wealth were chosen as variables of the dynamical system driven by the following set of equations

$$\frac{dA}{dt} = \mu\left[\alpha A - \beta A^2 - (\gamma - \delta)\,\frac{A}{a+A}\,\frac{I}{b+I}\right] \tag{2.1a}$$

$$\frac{dI}{dt} = -\xi I - \eta I^2 + \delta\,\frac{A}{a+A}\,\frac{I}{b+I}\,. \tag{2.1b}$$

Here μ represents the ratio of the productivities in agricultural over industrial sector. It is a scaling parameter. α represents the carrying capacity of the ecosphere and β defines the economies of size. a and b characterize economic recovery rates of agriculture and industry, respectively.

γ and δ are price indices of agriculture and industry, and their difference, $\gamma - \delta$, represents the terms of trade; ξ and η are linear and quadratic coefficients of the depreciation of the industrial wealth.

In practice all the parameters are positive constants, though it is supposed that in the more general theoretical case some of them, e.g., α, and ε may be negative depending upon the quality of the ecosphere.

Mathematical analysis conducted in [1] has shown that the model (2.1a) possesses an interior equilibrium that may be stable, or unstable and surrounded by a stable periodic trajectory (limit cycle). The latter occurs as a result of a Hopf bifurcation [2], when μ increases beyond a certain value known as the bifurcation value. Two other types of bifurcations may take place. Saddle-node bifurcation leads to the creation of two equilibria, one of them unstable, and the other stable at higher levels of A and I. The second type is a homoclinic bifurcation, that occurs when agricultural productivity increases beyond certain values determined for each combination of all other parameter values. This bifurcation leads to breaking the limit cycle and the emergence of the high level stable equilibrium as the only attractor of the system.

Notice here that all these features of the model (2.1) are inherent in the newest model when stabilization of the ecosphere is achieved.

In the later versions of the model [3,4] ecological wealth (E) is included as an additional dynamic variable. Parameter α determining the carrying capacity of the ecosphere is replaced by the function

$$\alpha(E) = \frac{\alpha E}{e + E}, \tag{2.2}$$

where α and e are constants. e is a new parameter characterizing the recovery rate of the ecosphere: the larger is e – the slower is recovery of the ecosphere (see Figure 2.1).

These two models represented two different approaches in the management of the sustainable agricultural systems, based upon the dynamics of the E-variable. In [3] the approach corresponds to the so-called Minimum Safe Standard approach. The equation determining the E-dynamics has the form

$$\frac{dE}{dt} = u(E - E_0) - vA(E - E_0) - w(E - E_0)^2. \tag{2.3}$$

The value of E cannot drop lower than E_0, the minimum safe standard, for any initial condition located at the level $E > E_0$.

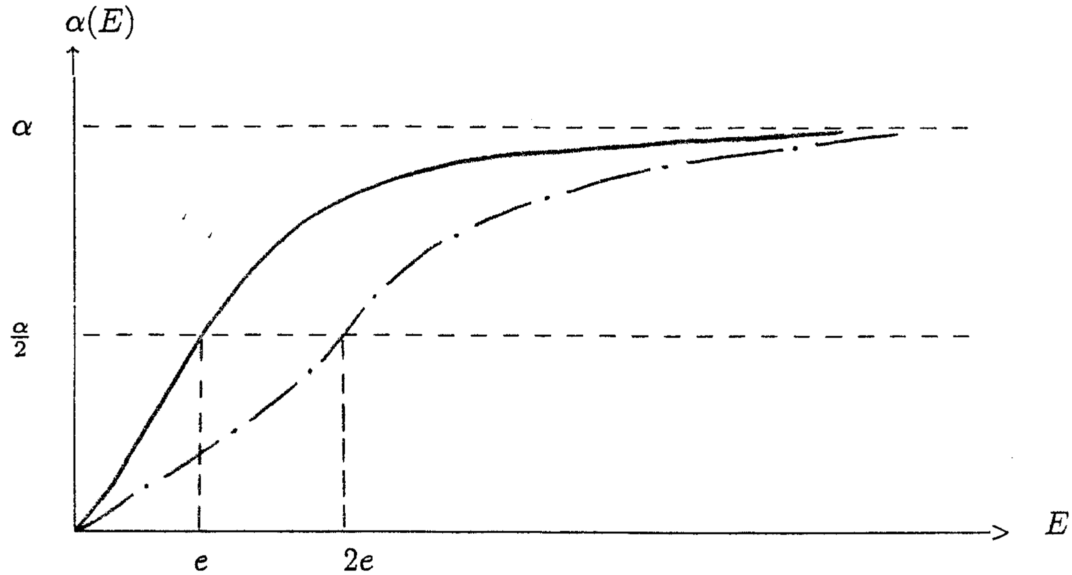

Figure 2.1

$$\text{———} \quad \alpha_1(E) = \frac{\alpha E}{e+E}$$

$$- \cdot - \cdot - \quad \alpha_2(E) = \frac{\alpha E}{2e+E}$$

It takes twice as much time for the lower graph to reach half-maximum value of $\alpha(E)$ comparing to the first graph. In that sense one can say that in case of $\alpha_1(E)$, the recovery is twice as fast as for $\alpha_2(E)$. The recovery rate is inversely proportional to e.

This is a regulatory approach to management of the environment. The results of our studies have shown that such an approach may lead to the appearance of a strange attractor [5] as a stable limit set of the system. The strange attractor eliminates predictability in the system: producing so-called deterministic chaos with consequent uninsurable uncertainty, raising serious questions about a regulatory approach to sustainability. In this mode the system's state at each single moment is defined uniquely by the equations (2.1-3), but the sensitivity to the initial conditions is extremely high. As in any real system its initial condition is always determined approximately. Therefore this sensitivity does not enable us to predict the system's state at any given moment of time.

The Recovery Model [4] replaces the threshold of a minimum safe standard by a natural recovery process. The evolution of E is described as

$$\frac{dE}{dt} = \varepsilon + uE - vAE - wE^2. \tag{2.4}$$

The ecosphere is also supported by external investments (term uE) and recovers naturally, but slowly, at a rate equal to ε. This small parameter represents endogenous biophysical recovery processes, such as improvement in soil organic matter, when $\varepsilon > 0$, or natural degradation, such as desertification, if $\varepsilon < 0$.

Solomonovich et al. has shown that in the case of $\varepsilon \geq 0$ the system possesses the property of persistence [6]. None of the dynamic variables of the systems vanish ensuring the sustainability of the agricultural system at some level.

The recovery model does not contain the prerequisites necessary for the formation of strange attractors. The most probable state of the system is oscillation of the dynamic variables with large amplitudes. This is not anymore of a desirable outcome than are strange attractors. The trajectories spend long periods of time when the values of A, I and E are all low because the velocity of the system's state in phase space is much slower near the origin, than at high values for the variables.

Thus, neither the Threshold Model [3] nor the Recovery Model [4] offer attractive ways to manage the development of the agricultural system and its relationships with the environment and the rest of the economy.

The purpose of this study is to construct a model of management that would provide for the sustainability of an agricultural system without the disadvantages of the preceding approaches.

2. Modifications and Their Interpretation

Let us turn to the "ecological" equation. We propose the following equation for rate of change of ecological wealth with respect to time:

$$\frac{dE}{dt} = \varepsilon - wE - vEA + u_1 A \left(1 - \frac{E}{e+E}\right). \tag{2.5}$$

As before, ε is a parameter representing natural recovery of the ecosphere; it is easy to see that in the absence of any agricultural activity in a system $(A \to 0)$ the ecological wealth tends to a stable value

$$\widetilde{E} = \frac{\varepsilon}{w}. \tag{2.6}$$

The term

$$u_1 A \left(1 - \frac{E}{e+E}\right) = u_1 A \, \frac{e}{e+E} \tag{2.7}$$

represents agricultural investment in the ecosphere. This investment constitutes a certain percentage of the total agricultural income per annum, so it is proportional to A. On the other hand, the lower the value of E, the greater the required investment. That is why the investment term is chosen to be proportional to the term

$$1 - \frac{E}{e+E}, \tag{2.8}$$

the difference between 1, the highest (asymptotical) level of the function representing the current state of the ecosphere, and the actual value of this function. When $E \to 0$, investment in rehabilitation takes its maximum value of $u_1 A$. When $E \to \infty$ investment tends to 0.

As this investment is financed from agricultural income, we have to introduce the corresponding correction in the equation (2.1a). Another correction introduced in this equation takes into account the dependence of net agricultural income, obtained from selling agricultural products, upon the quality of the ecosphere.

Finally, the equation replacing (2.1a) in the revised version of the model takes the form

$$\frac{dA}{dt} = \mu \left[\frac{\alpha E}{e+E} A - \beta A^2 - u\left(1 - \frac{E}{e+E}\right) + \left(\gamma \frac{E}{e+E} - \delta\right) \frac{A}{a+A} \frac{I}{b+I} \right], \tag{2.9}$$

where

$$\mu u \geq u_1, \tag{2.10}$$

meaning that investment in E cannot exceed the financing available from agricultural wealth.

There are also minor revisions in the equation describing the evaluation over time of industrial wealth. We allow an opportunity for external investment in industry. It is represented by a (small) parameter ζ. The range of ξ, a coefficient representing the linear part of the industrial cost function, is extended into the negative semiaxis.

Thus, after some simplifications in (2.5) and (2.9), the revised model is represented by the following dynamical system:

$$\frac{dA}{dt} = \mu \left[\frac{\alpha E - ue}{e+E} A - \beta A^2 + \left(\gamma \frac{E}{e+E} - \delta\right) \frac{A}{a+A} \frac{I}{b+I} \right] \tag{2.11a}$$

$$\frac{dI}{dt} = \zeta - \xi I - \eta I^2 + \delta \frac{A}{a+A} \frac{I}{b+I} \tag{2.11b}$$

$$\frac{dE}{dt} = \varepsilon - wE - vAE + u_1 A \frac{e}{e+E} \tag{2.11c}$$

with the initial conditions

$$A_0 > 0; \quad I_0 > 0; \quad E_0 > 0 \tag{2.12}$$

and some restrictions on the parameters.

Particularly:

(i) all parameters except ζ and ξ are positive;

(ii) the inequality $u_1 \leq \mu u$ (2.10) holds.

3. Representation of the Globalization Effect

In this study we are mostly concerned with local, comparatively small, systems of rural economies. Some of the parameters of these systems are subject to exogenous influences affecting the systems' performance. We refer to these influences as effects of globalization. For short periods of time we may consider all values of the parameters of our system (2.11) to be approximately constant. In the long run, at least, some of our system's parameters should be allowed to change especially under circumstances of trade liberalization and global technology transfer.

a) Agricultural and industrial productivities. Parameter μ represents the ratio of agricultural to industrial productivity of labors. Both productivities change with time in a non-uniform way to represent the ebb and flow of motivation to adopt new technology. These changes are approximated by the oscillation of the magnitude of their ratio. Modelling of $\mu(t)$ behaviour is a distinctive problem beyond the scope of our current research. The simplest starting point is to let the value of μ oscillate with time in accordance with the simplest sine-law:

$$\mu(t) = \mu\Big[1 + h \sin\Big(\frac{2\pi}{T}\, t\Big)\Big]. \tag{2.13}$$

h is the amplitude and T is the period of the oscillation.

b) Price indices and terms of trade. Price indices are changing exogenously following circumstances beyond the control of the local agricultural system. Statistical time series of market prices show that they oscillate. Both random and deterministic processes in the global system are viewed here as responsible for these oscillations. We model them by assigning a simple oscillatory behaviour with the period T and amplitude g to the parameter γ representing agricultural prices:

$$\gamma(t) = \gamma\Big[1 + g \sin\Big(\frac{2\pi}{T}\, t\Big)\Big]. \tag{2.14}$$

The effects of these exogenously generated oscillations on the systems' behaviour are explained in Section V of the paper. In the meantime, we assume that all other parameters of system (2.11) are constants.

III. Basic Mathematical Properties of the Model

1. Dissipativity

We are starting this section by proving that our system (2.11) is dissipative. Why is this important? Dissipativity is a necessary condition for persistence, and, consequently, sustainability of the system. We also roughly estimate the region of eventual system boundedness.

Proposition 3.1. *System* (2.11) *is dissipative with the attracting region contained in* $\mathcal{A} = \big\{(A, I, E) :\ 0 \le A \le \frac{1}{2}\big[\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + 4\max(\gamma - \delta, 0)}\,\big],\ 0 \le I \le \frac{\delta+\zeta}{\xi},\ 0 \le E \le \max\big(\frac{e}{2} + \sqrt{\frac{e^2}{4} + \frac{u_1 e}{v}},\ \frac{\varepsilon}{w}\big)\big\}$.

Proof. The part of the proof concerning A and I-variables is very similar to the proof of Theorem 2.1 in [4].

As for the E-variable, let us rewrite the corresponding equation (2.11c) as

$$\frac{dE}{dt} = \varepsilon - wE - A\psi(E), \tag{3.1}$$

where

$$\psi(E) = vE - \frac{u_1 e}{e+E} = \frac{v}{e+E}\Big(E^2 + eE - \frac{u_1 e}{v}\Big). \tag{3.2}$$

It follows from (2.16) that

$$\psi(E) \leq 0 \quad \text{for} \quad 0 \leq E \leq \frac{e}{2} + \sqrt{\frac{e^2}{4} + \frac{u_1 e}{v}} \tag{3.3a}$$

and

$$\psi(E) > 0 \quad \text{for} \quad E > \frac{e}{2} + \sqrt{\frac{e^2}{4} + \frac{u_1 e}{v}}. \tag{3.3b}$$

Then, if (2.17b) takes place,

$$\frac{d}{dt} E \leq \varepsilon - wE. \tag{3.4}$$

Introducing a new variable $\mathcal{E}(t)$ such that

$$\frac{d\mathcal{E}}{dt} = \varepsilon - w\mathcal{E} \tag{3.5}$$

and solving (3.5) with the initial condition $\mathcal{E}(0) = \mathcal{E}_0 \geq 0$, we obtain

$$\mathcal{E}(t) = \frac{1}{w}\left[\varepsilon + (w\mathcal{E}_0 - \varepsilon)e^{-wt}\right], \tag{3.6}$$

whence follows

$$E(t) \leq \frac{\varepsilon}{w}. \tag{3.7}$$

Thus, at least one of the inequalities (3.3a) or (3.7) is valid. The latter completes the proof.

2. Equilibria and their Stability

In this section we attempt to describe possible configurations of equilibria of system (2.11) together with their stability properties.

Equilibria of a dynamical system are solutions of the system that do not change with time. In our case that means

$$\frac{dA}{dt} = \frac{dI}{dt} = \frac{dE}{dt} = 0. \tag{3.8}$$

Thus the equilibria are determined by the system

$$A\Big[\frac{\alpha E}{e+E} - \beta A - \frac{ue}{e+E} + \Big(\gamma\,\frac{E}{e+E} - \delta\Big)\,\frac{A}{a+A}\cdot\frac{I}{b+I}\Big] = 0 \tag{3.9a}$$

$$\zeta - \xi I - \eta I^2 + \delta\,\frac{A}{a+E}\,\frac{I}{b+I} = 0 \tag{3.9b}$$

$$\varepsilon - wE - vAE + u_1 A\,\frac{e}{e+E} = 0. \tag{3.9c}$$

Local stability properties of equilibria are determined by the signs of the real parts of the eigenvalues of the variational matrix (Jacobian) of the system

$$V = \frac{\partial(\dot{A}, \dot{I}, \dot{E})}{\partial(A, I, E)} \tag{3.10}$$

For the system (2.11)

$$\begin{bmatrix} \mu\Big[\frac{\alpha E - ue}{e+E} - 2\beta A + \Big(\frac{\gamma E}{e+E} - \delta\Big)\frac{aI}{(a+A)^2(b+I)}\Big] & \mu\Big(\frac{\gamma E}{e+E} - \delta\Big)\frac{Ab}{(a+A)(b+I)^2} & \frac{\mu A e}{(e+E)^2}\Big[\alpha + u + \frac{\gamma I}{(a+A)(b+I)}\Big] \\ \delta\,\frac{aI}{(a+A)^2(b+I)} & -\xi - 2\eta I + \frac{\delta b A}{(a+A)(b+I)^2} & O \\ -vE + \frac{u_1 e}{e+E} & O & -w - vA - \frac{u_1 e A}{(e+E)^2} \end{bmatrix} \tag{3.11}$$

and for each given equilibrium, eigenvalues determining local stability properties are found from the characteristic equation

$$\det\,(V - \lambda I) = 0. \tag{3.12}$$

Here V is matrix (3.11) calculated at the given equilibrium. λ denotes the eigenvalues and I stands for the identity matrix.

As in our previous papers, we denote axial equilibria by $F^*(A^*, I^*, E^*)$ and planar equilibria by $\widehat{F}(\widehat{A}, \widehat{I}, \widehat{E})$.

a) Boundary equilibria.

Equilibria on axes

(i) A-axis: $I = 0,\ E = 0.$

(3.9) reduces to

$$\begin{cases} -u = 0 \\ \zeta = 0 \\ \varepsilon + u_1 A = 0. \end{cases}$$

This system does not have nonnegative solutions so there are no equilibria on the A-axis.

(ii) I-axis: $A = 0;\ E = 0.$

System (3.9) reduces to

$$\begin{cases} \zeta - \xi I - \eta I^2 = 0 \Longrightarrow I^* = \frac{-\xi \pm \sqrt{\xi^2 + 4\eta\zeta}}{2\eta} \\ \varepsilon = 0. \end{cases} \tag{3.13}$$

Hence, there is a nonnegative equilibrium

$$F^*\Big(0, \frac{-\xi + \sqrt{\xi^2 + 4\eta\zeta}}{2\eta}, 0\Big) \tag{3.14}$$

if and only if

$$\varepsilon = 0 \qquad \text{and} \quad \zeta \geq -\frac{\xi^2}{4\eta}\,. \tag{3.15}$$

Condition (3.15) is nonfeasible, and so this case is beyond our interest; still for mathematical completeness we should mention that V at F^* reduces to

$$V^* = \begin{pmatrix} -\mu\Big(u + \frac{\delta}{a} \cdot \frac{I^*}{b+I^*}\Big) & 0 & 0 \\ \frac{\delta}{a}\,\frac{I^*}{b+I^*} & -\xi - 2\eta I^* & 0 \\ u_1 & 0 & -w \end{pmatrix} \tag{3.16}$$

and all the corresponding eigenvalues are negative:

$$\lambda_1 = -\mu\left(u + \frac{\delta}{a}\,\frac{I^*}{b+I^*}\right); \quad \lambda_2 = -\xi - 2\eta I^*; \quad \lambda_3 = -w. \tag{3.17}$$

Consequently, this equilibrium is locally asymptotically stable (if it exists).

(iii) E-axis : $A = 0;\ I = 0.$

System (3.9) reduces to

$$\zeta = 0 \tag{3.18}$$

$$\varepsilon - wE = 0 \tag{3.19}$$

and there is an equilibrium

$$F^*\left(0,\ 0,\ \frac{\varepsilon}{w}\right) \tag{3.20}$$

if and only if (3.18) is satisfied.

In that case

$$V^* = \begin{pmatrix} \mu\,\frac{\alpha E^* - ue}{e+E^*} & 0 & 0 \\ 0 & -\xi & 0 \\ -vE^* + \frac{u_1 e}{e+E^*} & 0 & -w \end{pmatrix} \tag{3.21}$$

and the eigenvalues are

$$\lambda_1 = \mu\,\frac{\alpha E^* - ue}{e+E^*}; \quad \lambda_2 = -\xi < 0; \quad \lambda_3 = -w < 0. \tag{3.22}$$

$$\alpha E^* - ue = \alpha\,\frac{\varepsilon}{w} - ue.$$

Thus $\lambda_1 > 0$ if

$$\alpha\,\frac{\varepsilon}{w} - eu > 0 \tag{3.23}$$

and is negative or zero otherwise.

Let us notice that for the range of structural parameters we work in, inequality (3.23) is always satisfied. Therefore F^* defined by (3.20) is a saddle point, unstable locally in the A-direction.

Equilibria in the coordinate planes

(i) $I - E$ plane: $A = 0$.

System (3.9) reduces to the system

$$\zeta - \xi I - \eta I^2 = 0 \tag{3.24a}$$

$$\varepsilon - wE = 0 \tag{3.24b}$$

which possesses one and only one positive solution

$$\widehat{I} = \frac{-\xi + \sqrt{\xi^2 + 4\eta\zeta}}{2\eta} \tag{3.25a}$$

$$\widehat{E} = \frac{\varepsilon}{w}. \tag{3.25b}$$

The local stability properties of the equilibrium

$$\widehat{F}\Big(0, \frac{-\xi + \sqrt{\xi^2 + 4\eta\xi}}{2\eta}, \frac{\varepsilon}{w}\Big) \tag{3.26}$$

are determined by the matrix

$$\widehat{V} = \begin{pmatrix} \mu\left[\frac{\alpha\widehat{E} - ue}{e+\widehat{E}} + \left(\frac{\gamma\widehat{E}}{e+\widehat{E}} - \delta\right)\frac{\widehat{I}}{a(b+\widehat{I})}\right] & 0 & 0 \\ \frac{\delta}{a}\,\frac{\widehat{I}}{b+\widehat{I}} & -\xi - 2\eta\widehat{I} & 0 \\ -vE^* + \frac{u_1 e}{e+\widehat{E}} & 0 & -w \end{pmatrix}. \tag{3.27}$$

The eigenvalues are equal to the corresponding diagonal elements of $\widehat{V}$, so that the equilibrium is locally asymptotically stable in the I and E-directions.

This equilibrium should be unstable in the A-direction to ensure persistence, or sustainability, of the agricultural system.

Thus we subject the parameters of the system to the condition

$$\frac{\alpha\widehat{E} - ue}{e + \widehat{E}} + \left(\frac{\alpha\widehat{E}}{e + \widehat{E}} - \delta\right)\frac{\widehat{I}}{a(b + \widehat{I})} > 0, \tag{3.28}$$

and $\widehat{F}$ defined by (3.26) is a saddle point repelling trajectories from the plane $A = 0$.

(ii) $A - I$ plane: $E = 0$.

System (3.9) reduces into the system

$$-\beta A - u - \delta\frac{AI}{(a + A)(b + I)} = 0$$

$$\zeta - \xi I - \eta I^2 + \delta\frac{AI}{(a + A)(b + I)} = 0$$

$$\varepsilon + u_1 A = 0$$

which has no solutions for positive A and I. Hence, there are no equilibria in the $E = 0$ plane.

(iii) $A - E$ plane: $I = 0$.

(3.9) reduces to

$$\frac{\alpha E - ue}{e + E} - \beta A = 0 \tag{3.29a}$$

$$\zeta = 0 \tag{3.29b}$$

$$\varepsilon - wE - vAE + u_1 A\frac{e}{e + E} = 0. \tag{3.29c}$$

It follows from this system that there are no equilibria in the A, E plane when $\zeta \neq 0$. Now consider the case when $\zeta = 0$, meaning the absence of external investments in the system's industrial component. In this case we prove that (3.29a) and (3.29c) have simultaneous solutions. Each of these represents an equilibrium in the $A - E$ plane.

Proposition 3.2. *System* (2.11) *under condition* (3.29b) *possesses exactly one positive equilibrium if*

$$e > 1, \quad \varepsilon = O(w), \quad u_1 = O(v). \tag{3.30}$$

Otherwise it possesses at most three equilibria. In either case the equilibria are repellers locally in the I*-direction if* $\delta > b\xi$.

Proof. We start with the proof of the existence of at least one equilibrium.

If we denote

$$\varphi_1(E) \equiv \frac{1}{\beta}\,\frac{\alpha E - ue}{e + E}, \tag{3.31a}$$

$$\varphi_2(E) \equiv \frac{\varepsilon - wE}{vE - \frac{u_1 e}{e+E}}, \tag{3.31b}$$

we can rewrite system (3.29) as

$$A = \varphi_1(E) \tag{3.32a}$$

$$A = \varphi_2(E). \tag{3.32b}$$

That means that our equilibria (if they exist) are the points of intersections of the graphs of the functions φ_1 and φ_2 in the A, E-plane. Let us analyze the behaviour of these functions. Note, that we are considering only nonnegative values of E.

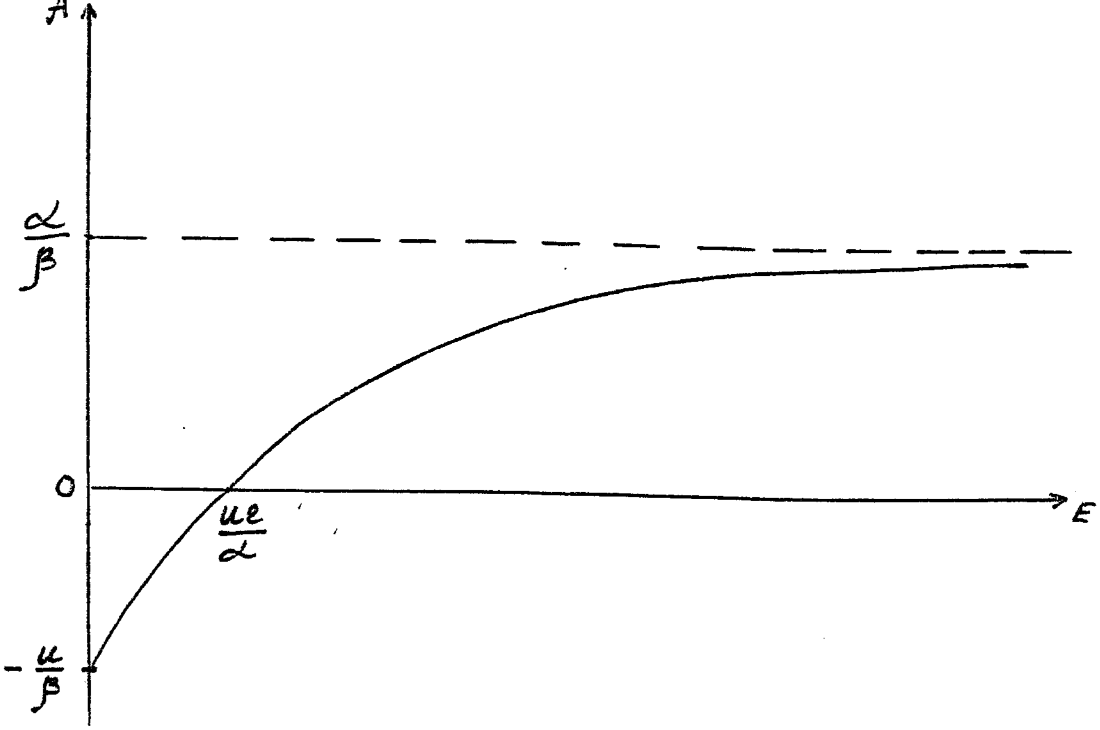

Figure 3.1. Graph of $A = \varphi_1(E)$

It is easy to see that $\varphi_1(E)$ is monotonically increasing, always concave down function with intercepts

$$\varphi_1(0) = -\frac{u}{\beta} \quad \text{and} \quad \varphi_1\left(\frac{eu}{\alpha}\right) = 0 \tag{3.33}$$

and a horizontal asymptote

$$A = \frac{\alpha}{\beta} \quad \text{when} \quad E \to +\infty.$$

The graph of $\varphi_1(E)$ is presented in Figure 3.1.

Function $\varphi_2(E)$ has a vertical asymptote at

$$E = -\frac{e}{2} + \sqrt{\frac{e^2}{4} + \frac{eu_1}{v}} \equiv E_V \tag{3.34}$$

and the graph of the function intersects the E-axis at

$$E = \frac{\varepsilon}{w} \equiv E_i. \tag{3.35}$$